ON THE QUESTION OF WHETHER ALL THE ROOTS OF AN ALGEBRAIC EQUATION HAVE A NEGATIVE REAL PART (STABILITY PROBLEM)

Hermann Schmidt ZAMM- 11/12, 1950- Vol. 30

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May 1, 1951

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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The different algebraic and geometric stability criteria presuppose particular algebraic and function-theoretic knowledge with which every engineer is not familiar. Besides, the relation between the application of these criteria and their derivation is not so simple that it is possible to have this relation present during its application. For the following simply established stability criterion, the contribution is easily realized that each step of the calculation answers the stability question.

The close connection of this criteria with the criterion of J. Schur is given at the end.

We limit ourselves to the most important engineering case of positive real coefficients and introduce the positive real parameter λ_1 into the polynomial equation

$$\Delta(p) = g(p) + h(p) = 0 \qquad (1)$$

formed from

$$g(p) = c_0 + c_2p^2 + c_4p^4 + \ldots$$

and

$$h(p) = c_1p + c_3p^3 + c_5p^5 + \ldots$$

in which we multiply $h(p)$ by λ_1.

We obtain

$$\Delta(p) = g(p) + \lambda_1 h(p) = 0 \tag{2}$$

and state:

If $\Delta p = 0$ has

N_1 roots with positive real part $(R > 0)$

N_2 roots with negative real part $(R < 0)$

N_3 roots on the imaginary axis $(R = 0)$

then $\Delta\lambda_1(p)$ has just so many roots in each of the two half-planes and on the imaginary axis. By means of the introduction of λ_1, no roots can leave the imaginary axis or cross or move on the imaginary axis. This follows simply from the theorem that the position of the intersection of the root curves

$$\begin{aligned} P(x,y) &= 0 \\ Q(x,y) &= 0 \end{aligned} \tag{3}$$

of

$$(p) = P(x,y) + iQ(x,y) = 0 \qquad p = x + iy \quad .$$

with the imaginary axis is independent of $\lambda_1 > 0$.

These intersection points y are yielded as roots of equation (3) for $x = 0$; the equations are

$$\begin{aligned} P_o(y) &= c_o - c_2y^2 + c_4y^4 - \dots = 0 \\ Q_o(y &= c_1y_1 - c_3y^3 + c_5y^5 - \dots = 0 \end{aligned} \tag{4}$$

By this means, the equations of the root curves for $\Delta\lambda_1(p)$ differ from those for $\Delta(p)$ only in that $Q_o(y)$ is multiplied by λ_1. The root curves of $\Delta(p)$ and $\Delta\lambda_1(p)$ therefore cut the imaginary axis in the same points as those denoted in the representation of the first example many times. N_3 therefore has the same value for equations (1) and (2).

Since further the roots of (1) are continuous functions of the equations of the coefficients then no real part of a root of $\Delta(p)$ can change its sign because of the introduction of λ_1. Namely if a root can cross the imaginary axis, then it must also give a value of λ_1 for which R=0. This is impossible, however, since λ_1 leaves invariant the intersection of the root curves with the imaginary axis.

For the answer to the stability question, therefore, (1) can be replaced by (2). Now, in (2) set

$$\lambda_1 = -\frac{g(p_1)}{h(p_1)} = -\frac{c_0+c_2p_1^2+c_4p_1^4+\dots}{c_1p_1+c_3p_1^3+\dots}$$

where p_1 is a negative real, for example, $p_1 = -1$; then

$$\Delta\lambda_1(p) = g(p)\cdot h(p_1) - g(p_1)h(p) = 0 \qquad (5)$$

This expression vanishes for $p = p_1$; it is therefore divisible by $p - p_1$, so that we obtain the equation of (n-1)st degree

$$\begin{aligned}\Delta_1(p) &= \frac{g(p)h(p_1)-g(p_1)h(p)}{p-p_1} \\ &= g_1(p)+h_1(p) = 0\end{aligned} \qquad (6)$$

that has in the negative half-plane one root less than the initial equation; we have to force it by means of a choice of the parameter λ_1 nearest to the root p_1, which we have then split off.

If (6) produces a sign change then one must proceed with (6) as with (1) whereby, however, one can split off also another arbitrary real negative root of p_1 $(p_1 \neq -1)$. One thus obtains an equation of degree $n-2$ with the number of roots N_3, N_{n-2}, N_1. Therefore, one has to form

$$\Delta_{\lambda_2}(p) = g_1(p) + \lambda_2 h_1(p) \tag{7}$$

with

$$\lambda_2 = - \frac{-g_1(p_1)}{h_1(p_1)}$$

and from (7)

$$\Delta_2(p) = \frac{\Delta_{\lambda_2}(p)}{p-p_1} = g_2(p) + h_2(p) = 0 \tag{8}$$

The signs of $\Delta_2(p)$ determine the necessity of the formation of $\Delta_3(p)$.

These processes of reducing the degree of the preceding equation is continued as long as no sign change occurs in the reduced equation, at most until an equation of second degree which is reached after $n-2$ steps. If all the reduced equations are free of sign changes, then $N_1 = 0$; since N_3 is not changed by this procedure then the sequence of equations $\Delta_v(p)$ ends in that equation which has only the pure imaginary roots of $\Delta(p)$, if $\Delta(p)$ possesses such roots. In general, it is appropriate to set $p_1 = -1$ for all the reduction steps, so that

$$\lambda = \frac{c_0+c_2+c_4+\ldots}{c_1+c_3+c_5+\ldots}$$

The necessary divisions by $(p+1)$ are carried out by Horner's method.

Schur [1] classified the proposed polynomial $\Delta(p)$ of the n-th degree with another $\Delta^*(p)$ of the same degree, the roots of which are mirror-images of the roots of $\Delta(p)$ with respect to the imaginary axis; he uses the expression already applied by Hermite [2] for similar investigations

$$F(p,p_1) = \Delta^*(p_1)\Delta(p) - \Delta^*(p)\Delta(p_1) = 0 \qquad (9)$$

for the reduction process. $F(p,p_1)$ is divisible by $p-p_1$.

If, for real coefficients, one sets

$$\Delta(p) = \alpha g(p) + \beta h(p)$$

$$\Delta^*(p) = \alpha_1 g(p) + \beta_1 h(p)$$

then

$$F(p,p_1) = \begin{vmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{vmatrix} \left[g(p)h(p_1) - g(p_1)h(p) \right]$$

or

$$F(p,p_1) = h(p_1) \begin{vmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{vmatrix} \left[g(p) - \frac{g(p_1)}{h(p_1)} h(p) \right]$$

Expression (9) therefore corresponds to (2) or (5) respectively, except for a constant factor.

First Example: Stable Case

Let the assumed equation be

$$\Delta(p) = 8p^6 + 13p^5 + 120p^4 + 137p^3 + 450p^2 + 250p + 322 = 0$$

For $p_1 = -1$,

$$\lambda_1 = \frac{322+450+120+8}{250+137+13} = 2.25$$

and $\Delta_{\lambda_1}(p) = 8p^6 + 29.25p^5 + 120p^4 + 308.25p^3 + 450p^2 + 562.5p +$

$322 = 0$

As the figure of the root curves also shows, $\Delta\lambda_1(p)$ - and especially $\Delta\lambda_2(p)$, $\Delta\lambda_3(p)$ and $\Delta\lambda_4(p)$ - has the forced root $p_1 = -1$; by means of the introduction of λ_1 an original pair of complex roots degenerates into two real roots of which one (p_1) can be split off by means of Horner's method.

The method is

	8	29.25	120	308.25	450	562.5	322
-1		-8	-21.25	-98.75	-209.5	-240.5	-322
	8	21.25	98.75	209.5	240.5	322	

Therefore the first reduced equation is

$$\Delta_1(p) = 8p^5 + 21.25p^4 + 98.75p^3 + 209.5p^2 + 240.5p + 322 = g_1(p) + h_1(p) = 0$$

For $\lambda_2 = 1.59$

$$\Delta_{\lambda_2}(p) = 5896p^5 + 9838.75p^4 + 72778.75p^3 + 96998.65p^2 + 177248.5p + 149086 = 0$$

and

$$\Delta_2(p) = 5896p^4 + 3942.75p^3 + 68836p^2 + 28162.5p + 149086 = 0$$

Similarly, for $\lambda_3 = 6.97$

$$\Delta_3(p) = p^3 + 3.662p^2 + 8.013p + 25.286 = 0$$

and for $\lambda_4 = 3.212$

$$\Delta_4(p) = p^2 + 0.140p + 7.873 = g_4(p) + h_4(p) = 0$$

Since in the series of $\Delta_v(p)$ none of these equations shows a sign change, then $\Delta(p)$ has no roots $R > 0$; since further, $\Delta_4(p)$ has no pure imaginary roots the limit case of stability is not discussed. Therefore, $\Delta(p)$ has only roots $R < 0$.

Second Example: Limit case of stability

$$\Delta(p) = 6\,p^7 + 4p^6 + 8p^5 + 12p^4 + 13p^3 + 12p^2 + 6p + 4 = 0$$

reduces for $\lambda_1 = 8/7$; $\lambda_2 = 9/5$; $\lambda_3 = 7/2$ to the reduced equations

$$\Delta_1(p) = 8p^6 + 20p^5 + 44p^4 + 40p^3 + 64p^2 + 20p + 28 = g_1(p) + h_1(p) = 0$$

$$\Delta_2(p) = 40p^5 + 140p^4 + 80p^3 + 280p^2 + 40p + 140 = g_2(p) + h_2(p) = 0$$

$$\Delta_3(p) = 28p^4 + 56p^3 + 28 = g_3(p) + h_3(p) = 0$$

with the double-root-pair $p = \pm i$ which is also a root-pair of $\Delta(p)$.

Third Example: Unstable case

The example given by Schur [1] is

$$\Delta(p) = 8p^5 + 4p^4 + 7p^3 + 5p^2 + p + 1 = 0$$

For $\lambda_1 = 5/8$ we obtain

$$\Delta_{\lambda_1}(p) = 40p^5 + 32p^4 + 35p^3 + 40p^2 + 5p + 8 = 0$$

Horner's method

	40	32	35	40	5	8
-1		-40	8	-43	3	-8
	40	-8	43	-3	8	

yields, in its last line, the coefficients of $\Delta_1(p)$ with a sign change. Therefore, it produces instability.

1. J. Schur: Uber algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen. ZAMM 1 (1921) p. 307-311

2. Ch. Hermite: Extrait d'une lettre de M. Ch. Hermite. Crelles Journal 52, 1854, pp. 39-51

Also- M. Bocher, Introduction to Higher algebra, 1910